How to apply a plaster on a drum to make it harmonic

S. Maugeais
LMM, Département de mathématiques, Avenue Olivier Messiaen, BP 535, 72017 Le Mans, France
sylvain.maugeais@univ-lemans.fr

It was known in Indian antiquity that adding a plaster to a drum reduced "the harshness of sound", meaning it produced an almost harmonic sound (cf. [Bha51], chapter 33, verse 25-26). This was observed experimentally for the tabla by Raman in the 1920's (cf. [RK20] and [Ram34]). It was also known in ancient India that it required a skilled musician to apply the plaster to improve the harmonicity. Originally, this plaster was temporary, and made out of clay. Nowadays, it is replaced on some instruments, such as the tabla, by a permanent application obtained with many layers of "syahi masala".
In this paper, an analysis of the plaster effect on the harmonicity of the instrument is proposed. For this purpose, a tabla membrane having a varying mass density is modeled: using a perturbation of the density on the wave equation of the membrane, we compute the changes in the eigenfrequencies when adding a thin circular mass at the center of the membrane. An optimization procedure based on the gradient algorithm is proposed for reaching harmonic eigenfrequencies for the non-homogenous membranes by applying many such layers. Finally, this algorithm is tested for different initial densities with and without additional constraints.

1 The composite membrane of the tabla

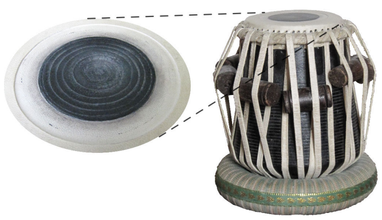

Figure 1: The *syahi* of a tabla

According to Courtney (cf. [Cou01], page 94), the *syahi* of the tabla is obtained by the application of many layers of a paste made out of *syahi massala*: a powder containing soot and iron dust, water and flour (according to [FD98], chapter 18.5, the flour is replaced by overcooked rice for the *mrdanga*).

Figure 2: The reticulum of cracks on the tabla

Each layer is then rubbed with a stone so as to produce the characteristic cracks (cf. Figure 2) thus reducing the rigidity of the *syahi* and letting the membrane vibrate more freely. That part of the process will not be considered as we are only dealing with the geometry of the *syahi*. However, the rubbing process imposes special constraints for the density of the plaster: it should be smooth and decreasing with the radius. These will be important conditions when trying to optimize the density.

The number of layers and their geometry is crucial to obtain a good *syahi* (cf. Figure 3). There should be between 35 and 40 layers for the tabla (cf. [Cou01], page 95) and around a hundred for a mrdanga (cf. [FD98], chapter 18.5).

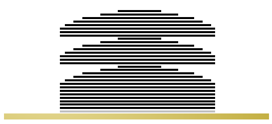

Figure 3: Cross section of a syahi, cf. [Cou01]

Figure 4 shows the spectrum of a tabla for different strokes, the horizontal dotted line representing the maxima of each stroke minus 3dB. The most important stroke while playing is *Na*. This is also the one used to tune the tabla and is characterized by the "absence" of fundamental: to play it, one finger is touching the skin while another one strikes the membrane at 90° to it on the rim of the membrane. As the fundamental is known to be off[1] this is the stroke that produces the spectrum closest to harmonic. As a consequence, the second overtone will always be taken equal to 2 as a reference.

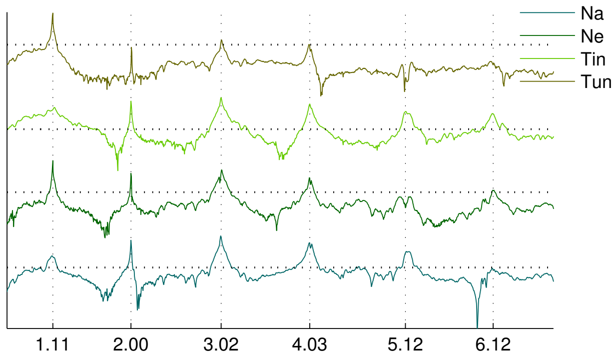

Figure 4: Spectrum of a tabla for different strokes

Table 1 shows the actual frequencies of a uniform membrane, a tabla in normal use and a tabla with opened shell (cf. [RS54]).

Table 1: Frequency ratios of a uniform membrane, a tabla in normal use and a tabla with no wooden shell, cf. [RS54]

Modes	uniform membrane	Tabla[a] in normal use	Tabla[a] with wooden shell opened out
(0, 1)	1.00	1.10	1.03
(1, 1)	1.59	2.00	2.00
(2, 1)	2.14	3.00	3.00
(0, 2)	2.30	3.00	3.00
(3, 1)	2.65	4.03	4.00
(1, 2)	2.92	4.01	4.00
(0, 3)	3.60	4.83	5.04
(4, 1)	3.16	5.07	5.03
(2, 2)	3.50	5.07	5.08

[a] second harmonic taken as 2 for reference

Following [GGL06] and the spectra shown in Figure 4, we added to our computations the mode $(1, 3)$ whose frequency should be about 6 times the fundamental. This choice proved to be crucial for the convergence of our method (see section 3.6).

[1] in [Cou01], page 96, Courtney says that the relation between the fundamental and the second harmonic should be a minor seventh, which is actually what is measured in Figure 4.

2 Mathematical model

Let us model our circular membrane by a disk D of radius 1 centered at $(0, 0)$, and a in-homogeneous density given by a function $\rho \in \mathcal{L}$ where

$$\mathcal{L} = \left\{ \rho \in L^2(D) |\, \rho \text{ and } \frac{1}{\rho} \text{ are uniformly bounded} \right\}.$$

As this paper is only concerned with the harmonicity of the overtones, we can take 1 for the radius of the membrane and for the minimum of the density.

Consider the Hilbert space $L^2(D, \rho)$ given by the scalar product $\langle f, g\rangle_\rho = \int_D \rho f g$. For $\rho = 1$, we can drop the index and obtain the usual scalar product on $L^2(D)$.

The operator $-\frac{1}{\rho}\Delta$ is defined on $\mathrm{H}^2(D) \cap \mathrm{H}^1_0(D)$ and is self-adjoint for the scalar product $\langle , \rangle_\rho$ so that its eigenvalues are real. Moreover, as ρ and $\frac{1}{\rho}$ are uniformly bounded, it has a compact inverse so its eigenvalues form a discrete set.

Suppose that ρ is a function of r only. Separating the variables, we have to look for solutions of the form $u(\mathbf{x}) = \varphi_n(r)\cos(n\theta)$ where (r, θ) are the polar coordinates of $\mathbf{x}$ and $n \in \mathbb{N}$. Then there should be an $\omega \in \mathbb{R}$ such that φ_n is solution of the equation

$$r^2\varphi_n'' + r\varphi_n' + (\omega^2\rho(r)r^2 - n^2)\varphi_n = 0 \tag{1}$$

and satisfies $\varphi_n(1) = 0$. Then $-\omega^2$ is an eigenvalue of $-\frac{1}{\rho}\Delta$.

For fixed ρ and n, we will order the possible ω in increasing order, denoting them by $\omega_{n,m}(\rho)$, $m \geq 1$: the number m then corresponds to the number of radial anti-nodes of the function φ_n.

For symmetry reasons the first vibration modes of the membrane should be as in Figure 5. The number under each drawing represents the corresponding ratio of the frequency to the fundamental for a perfect tabla, the indexes of p corresponding to the numbering of the ω. The colors and type of lines used in Figure 5 will be fixed for each mode and used throughout the article.

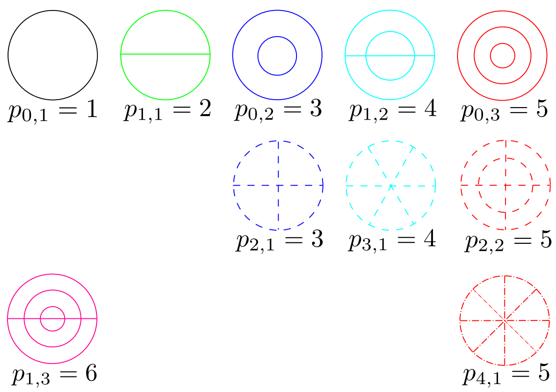

Figure 5: Vibration modes of a circular membrane and corresponding frequency ratios of a perfect tabla

When $\rho = 1$, $\omega_{n,m}(\rho)$ is the m-th zeros of the bessel function J_n and $\varphi_n(r) = J_n(r\omega_{n,m})$ is the corresponding solution.

In [RS54], Ramakrishna and Sondhi proposed a model for the tabla by studying a *composite membrane* composed of two domains:

$$\rho_{\text{comp}}(r) = \begin{cases} 9.765 \text{ if } r \leq 0.4 \\ 1 \text{ if } r > 0.4. \end{cases} \tag{2}$$

the quantity 9.765 being the optimal for the radius of a plaster equal to 40% of the total. Other models were considered, trying to refine this one, see for example [GGL06] who further studied the *composite membrane* and optimized the ratios of densities of the two domains together with the radius of the plaster, and [SA09] who used a continuous approximation of (2).

2.1 Perturbation of the density

Suppose that a density ρ is given and consider a small perturbation $\rho_\varepsilon = \rho + \varepsilon\tilde{\rho}$. We look for a solution $\varphi_\varepsilon = \varphi + \varepsilon\tilde{\varphi}$ for a parameter $\omega_\varepsilon = \omega + \varepsilon\tilde{\omega}$ satisfying $\varphi_\varepsilon(1) = 0$. Then, up to a power of ε^2, $\tilde{\varphi}$ must verify the equation

$$r^2\tilde{\varphi}'' + r\tilde{\varphi}' + (\omega^2r^2\rho - n^2)\tilde{\varphi} = -(2\omega\tilde{\omega}r^2\rho + \omega^2r^2\tilde{\rho})\varphi. \tag{3}$$

Looking for a solution of the form $\tilde{\varphi} = g\varphi$ and using variation of the parameters, we find that

$$g'(r) = \frac{-\omega}{r\varphi(r)^2}\int_0^r (2\tilde{\omega}\rho(s) + \omega\tilde{\rho}(s))s\varphi^2(s)ds \tag{4}$$

If it is supposed further that ρ and $\tilde{\rho}$ are C^1 in a neighbourhood of 1 (for example constants), then so are φ and $\tilde{\varphi}$. An analysis of the multiplicity of 1 as a zero of $g\varphi$ shows that $g'(1) = 0$, which means that $\tilde{\omega} = \langle \mathrm{grad}_\rho\,\omega, \tilde{\rho}\rangle$, where

$$\mathrm{grad}_\rho\,\omega = -\frac{\omega r\varphi^2(r)}{2\displaystyle\int_0^1 s\varphi^2(s)\rho(s)ds} \tag{5}$$

is the gradient of the functional ω as a variable of ρ, and $\langle ., . \rangle$ is the scalar product of $L^2([0, 1])$. Note that *a priori* this is actually the gradient only for a variable ρ and a perturbation $\tilde{\rho}$ that are C^1 in a neighbourhood of 1.

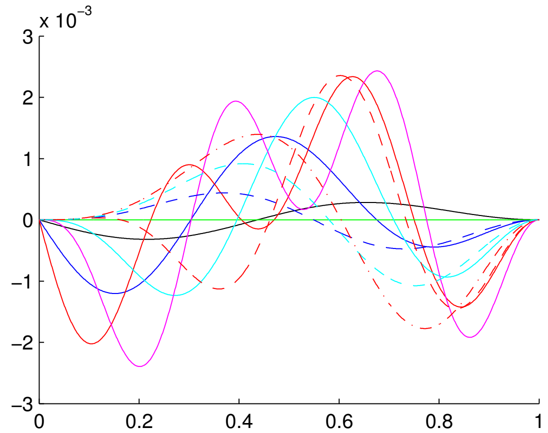

Figure 6: Gradient of the $\omega_{n,m}$ for uniform density, the colors follow those of Figure 5.

In particular, if one eigenvalue ω_{ref} is fixed for normalization, we find that

$$\frac{\omega_{n,m}(\rho_\varepsilon)}{\omega_{ref}(\rho_\varepsilon)} = \frac{\omega_{n,m}(\rho)}{\omega_{ref}(\rho)} + \varepsilon\left\langle \mathrm{grad}_\rho\left(\frac{\omega_{n,m}}{\omega_{ref}}\right), \tilde{\rho}\right\rangle \tag{6}$$

with

$$\mathrm{grad}_\rho\left(\frac{\omega_{n,m}}{\omega_{ref}}\right) = \frac{\mathrm{grad}_\rho\,\omega_{n,m}}{\omega_{ref}} - \frac{\omega_{n,m}}{\omega_{ref}^2}\mathrm{grad}_\rho\,\omega_{ref}. \tag{7}$$

2.2 Application to the tuning of tabla

Suppose that a thin layer of density ε and radius a is added to a membrane of density ρ, i.e. $\tilde{\rho} = \mathbf{1}_{[0,a[}$, then the equation (6) gives

$$\frac{\omega_{n,m}(\rho_\varepsilon)}{\omega_{1,1}(\rho_\varepsilon)} = \frac{\omega_{n,m}(\rho)}{\omega_{1,1}(\rho)} + \varepsilon \int_0^a \mathrm{grad}_\rho \frac{\omega_{n,m}}{\omega_{1,1}} dr. \tag{8}$$

Figure 7 represents these functions for $\rho = 1$ as a varies. The vertical dotted line represents the points where the modes corresponding to the third, fourth fifth and sixth harmonics go past the second one. Thus, for $a \geq 0.623$, a thin disk of radius a will increase the ratios $\frac{\omega_{n,m}}{\omega_{1,1}}$ for all $n \geq 2$.

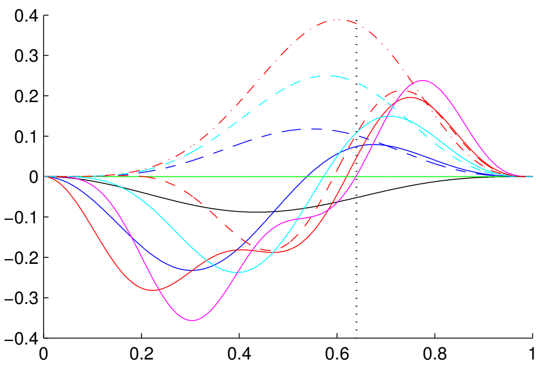

Figure 7: Effect of adding a layer of variable size to the uniform density

This pattern depends heavily on the density ρ and cannot serve as a simple guide for the tuning of tabla, as can be seen in Figure 8 where we show the same function but for

$$\rho(r) = \begin{cases} 1 \text{ if } r \leq 0.4 \\ 5 \text{ if } r > 0.4 \end{cases}$$

which represents a density between the uniform one and the optimal composite of [RS54].

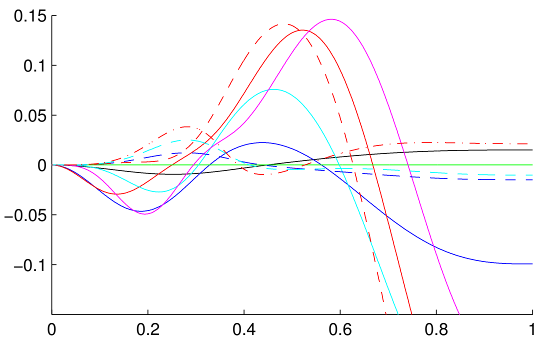

Figure 8: Effect of adding a layer of variable size to a composite density

3 Optimisation

We want to measure the distance between the spectrum of a disk with density ρ and the spectrum of a perfect tabla. Following Gaudet et al. in [GGL06], we define the functional

$$E : \rho \mapsto \sum_{n,m} \left(\frac{\omega_{n,m}(\rho)}{\omega_{1,1}(\rho)} - \frac{p_{n,m}}{p_{1,1}} \right)^2 \tag{9}$$

for $\rho \in \mathcal{L}$ where the number $p_{n,m}$ is given for the mode (n, m) in Figure 5. Table 2 gives the values of the function E for the different densities found in the literature.

Table 2: Value of the function E for the different solutions proposed in the literature

Uniform	[RS54]	[GGL06]	[SA09]
0.7253	0.0864	0.1983[a]	0.0234

[a] "bad" result owing to the fact that they took a frequency ratio to the fundamental of 6 for the mode $(0, 3)$ and 5 for the mode $(1, 3)$, and we did the opposite.

Our goal is then to find a minima of the function E. For this the gradient algorithm is used: at each step, the direction of steepest descent is given by the gradient of E. Using equation (7), we find that it is given by

$$\mathrm{grad}_\rho E = \sum_{n,m} 2 \left(\frac{\omega_{n,m}(\rho)}{\omega_{1,1}(\rho)} - \frac{p_{n,m}}{p_{1,1}} \right) \mathrm{grad}_\rho \left(\frac{\omega_{n,p}}{\omega_{ref}} \right). \tag{10}$$

Being given a function ρ, in order to apply a step of the gradient algorithm it is necessary to

- compute the eigenvalues of $\frac{1}{\rho}\Delta$, i.e. find the scalars ω such that there exists a function φ on $[0, 1]$ satisfying $\varphi(1) = 0$ and

$$r^2\varphi'' + r\varphi' - n^2\varphi = -\omega^2 r^2 \rho\varphi. \tag{11}$$

- compute the eigenvectors of $\frac{1}{\rho}\Delta$, i.e. find the solution φ corresponding to an ω in equation (11), and deduce the expression of $\mathrm{grad}_\rho E$ using equations (5), (7) and (10).
- find the scalar α that minimises $E(\rho - \alpha \mathrm{grad}_\rho E)$.

We then replace ρ by $\rho - \alpha \mathrm{grad}_\rho E$ and repeat the process until a minimum is found, or the precision of the computations is exceeded.

3.1 Eigenvalues of $\frac{1}{\rho}\Delta$

To compute the eigenvalues of $\frac{1}{\rho}\Delta$, a Finite Element (FE) scheme is usedThe weak formulation of equation (11) reduces to

$$\int_0^1 r^2\varphi'\psi' dr + \int_0^1 r\psi'\psi dr + n^2 \int_0^1 \varphi\psi dr = \omega^2 \int_0^1 r^2\varphi\psi dr \tag{12}$$

for all test function ψ.

Taking a uniform discretisation step h of $[0, 1]$ and order 1 simplicial elements in dimension 1 for the basis function, we have to solve the eigenvalue problem $A\Phi = \omega^2 B\Phi$ for square matrices A and B, where Φ is the vector of coordinates of φ in the basis of elements.

If ρ is C^2, the eigenvalues of $B^{-1}A$ converge towards the eigenvalues of $\frac{1}{\rho}\Delta$ as ch for a constant c. The results presented in Table 3, and computed for the uniform and composite density, correspond up to 3.10^{-3} for a discretisation step $h = 10^{-3}$.

Table 3: Comparison of theoretical results and computations by the finite element method

Modes	uniform		composite	
	Theory	FE	Theory	FE
(0, 1)	1.0000	1.0000	1.0000	1.0000
(1, 1)	1.5933	1.5933	1.9368	1.9361
(2, 1)	2.1355	2.1356	2.9470	2.9455
(0, 2)	2.2954	2.2954	3.0520	3.0511
(3, 1)	2.6531	2.6531	3.9666	3.9643
(1, 2)	2.9173	2.9173	4.0964	4.0956
(0, 3)	3.5985	3.5985	4.8198	4.8222
(4, 1)	3.1555	3.1555	4.9651	4.9621
(2, 2)	3.5001	3.5002	5.1448	5.1443

3.2 Eigenvectors of $\frac{1}{\rho}\Delta$

Although the finite element scheme converges well for eigenvalues, it is not the case for the eigenvectors. The reason for this is the presence of a singularity in the equation (11): the "mode" towards which the algorithm "converges" is actually an unbounded solution of the equation (11), which is useless in this case.

Thus we use the usual Euler method with constant step h (same as for the FE scheme) to find the solutions of (11) once ω is given. As the finite element scheme used to obtain the ω is only of order 1, there is no need to take a higher order method to solve the differential equation.

To get rid of the singularity, the problem is initialized at $r = 1$ imposing $\varphi(1) = 0$ and $\varphi'(1) = 1$ and compute only for $r \geq -n^2 h \log(h)$. The last part is interpolated by a polynomial of the form $r^{(n^2)}c$ as the equation (11) says that all the derivatives of φ_n at $r = 0$ of degree $< n^2$ are zero. In particular, it does not change the order of the method.

The solutions are computed for the composite membrane (ρ_{comp}) of [RS54] in Figure (9).

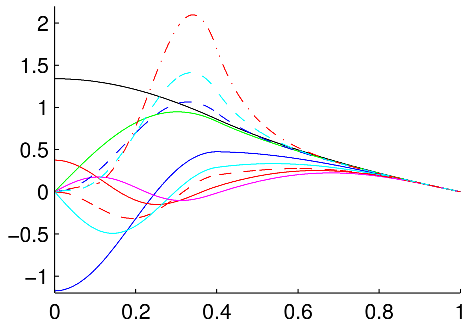

Figure 9: Eigenvectors for the different modes of the composite membrane of [RS54]

3.3 Line search

The direction of steepest descent now being known, we have to find the step α for which the quantity $E(\rho+\alpha\mathrm{grad}_\rho E)$ is minimal. This is done by a usual simplex method. This process is actually the most time-consuming, as it requires computing the function E for many different α. For this the $\omega_{n,m}$ are needed and they can only be computed by the finite element scheme described in section 3.1.

3.4 First results

Once the algorithm is ready, the first problem is to choose of the initial density as the algorithm does not converge for the obvious one (the uniform $\rho = 1$): for this choice of initialization, the algorithm gives a non-positive density after a few steps, and this does not verify our hypothesis of boundedness of $\frac{1}{\rho}$, and has no physical meaning.

The goal being to get the spectrum closest to harmonicity, the algorithm was initialized with the best solution so far, which is the continuous density of Sathej and Adikari (cf. [SA09]) for which $E(\rho_{cont}) = 0.0234$.

The algorithm stopped after 84 steps when it reached a minimum. Figure 11 shows different steps of the convergence of the density during the gradient algorithm

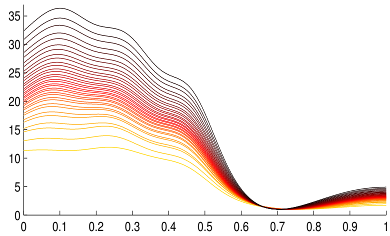

Figure 10: Convergence of the density during the gradient algorithm. The first steps are in yellow and the last in black

The numerical results are presented in Table 4 where the frequencies $\omega_{n,m}$ are added for $n, m \leq 5$. As can be seen in the table many other modes are close to integers.

Table 4: Normalised frequencies of the first modes, E=0.0015

$\omega_{n,m}$	m= 1	2	3	4	5
n=0	1.0713	2.9927	4.9985	7.0149	8.7165
1	2.0000	3.9880	6.0049	7.9196	9.4739
2	2.9852	5.0132	6.9260	8.2127	9.9873
3	3.9999	6.0509	8.1256	9.8502	11.0970
4	5.0192	7.0985	9.1989	10.8866	11.9717
5	6.0349	8.1463	10.2688	11.9514	12.8773

We see that all the frequencies > 2 are attained, but the frequency that is most difficult to get close to is 1: the algorithm does not allow us to lower the mode (0, 1) further.

Although the final solution has a very low value of E (equal to 0.0015), the density exceeds 35 at certain points and is not decreasing (see section 1), so it is not satisfactory.

3.5 Additional assumptions

Using the gradient algorithm without additional assumptions leads to unrealistic densities. The problem is that the functional E has many minima, including some for which the density may not be bounded, or may become singular.

We found that the following set of hypotheses, which are verified for the actual tabla, are actually enough to ensure the convergence to a physically realistic density:

- only add mass, and not remove any: the gradient $\mathrm{grad}_\rho E$ is replaced by the function $\min(0, \mathrm{grad}_\rho E)$;

- only add disks, without holes, i.e. ρ should be a decreasing function (see section 1 where this condition appeared because of the construction process): ρ is replaced by ρ_{dec} which is the smallest decreasing function that is $\geq \rho$;
- only add small layers: choose α such that $\max_r |\alpha \mathrm{grad}_\rho E| \leq 0.1$. This prevents the algorithm going directly to minima that are unbounded.

3.6 Final Results

The algorithm stops when it reaches a minimum, which was done in 57 steps much before the precision of the computation was exceeded (which is about 10^{-3}, as fixed by the FE scheme).

Figure 11 is obtained by initializing the algorithm at $\rho = 1$ and shows different steps for the convergence of the density. The final density is not too far off from the shape of an actual tabla, or even to the solution of [RS54].

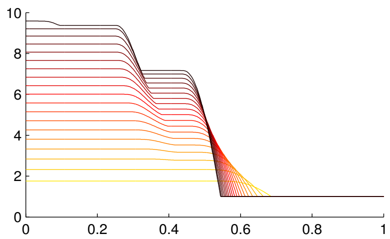

Figure 11: Convergence of the density during the gradient algorithm. The first steps are in yellow and the last in black

At the same time, Figure 12 shows how the frequencies change.

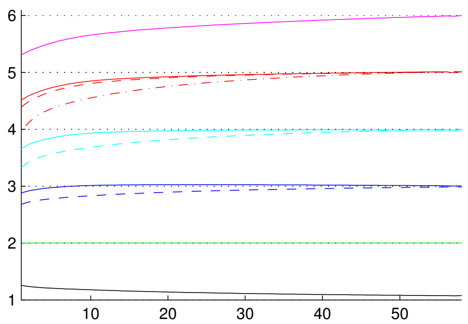

Figure 12: Convergence of frequencies during the gradient algorithm

As can seen in Table 5, many of the other modes (those that are not taken into account in the error functional E) are close to integers. The value of E for the final density is 0.0016, which is as good as the one obtained in section 3.4 without the extra conditions.

It should be noted here that the results depend heavily on the modes that we are considering: adding or removing some modes, or even adding some weights to them would produce a different result! In particular, we found that considering the mode $(1, 3)$ with ratio to the fundamental equal to 6 produced the result closest to reality.

Table 5: Normalised frequencies of the first modes, E=0.0016

$\omega_{n,m}$	m=1	2	3	4	5
n=0	1.0739	3.0065	5.0128	6.6834	8.1350
1	2.0000	3.9911	6.0026	7.3789	9.0474
2	2.9852	5.0132	6.9260	8.2127	9.9873
3	3.9966	6.0751	7.8538	9.1230	10.9558
4	5.0104	7.1633	8.8376	10.0568	11.9585
5	6.0144	8.2627	9.8896	11.0003	12.9856

4 Conclusion

Using a distance to harmonicity inspired by Gaudet et al. [GGL06] and by the actual spectra of the tabla, we were able to apply the gradient algorithm and find a density that optimizes the spectrum for inhomogeneous membrane. The results, obtained for a particular choice of important modes and extra assumptions modelling the actual construction are close to the heuristic model of [RS54], and even closer to the actual tabla.

The choice of the "important" modes was made by following the literature and the spectrum of a real tabla. It would be interesting to justify it by studying the damping of each mode in relation to the second skin and the shell of the tabla, and keeping only those that are prevalent.

Acknowledgements

The author would like to thank Simon Félix for advises on the bibliography, and François Gautier for discussions on the subject.

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