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”Résonance Trapping” dans un Guide Traité par une Impédance Locale

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Resonance trapping phenominon in an open quantum system has been recently observed in an open microwave cavity. With increasing coupling strength to the continuum of decay channels, only a few resonances align with the open channels and become short lived, while the widths of the remaining resonances first increase but finally decrease again and become long lived (trapped modes). In this paper, we present an analogy of resonance trapping in open quantum system to modes in an acoustic close system, a waveguide with impedance boundary conditions. Our results show that resonance trapping may take place not only in open systems as illustrated in the literature, but also in close systems. The important ingredient is the exist of Exceptional Points (EPs). Our analogy provides a nouvel insight into resonance trapping and provide a new point of view for understanding the mode behaviours in waveguides.

1 Introduction

Mode in an infinite Waveguide with Impedance Boundary Conditions (WIBC) is given, as a basic concept, in textbooks such as Refs. [1, 2]. It provides a deep understanding of the complex sound field in a waveguide. Mode propagation in a waveguide with finite length of impedance boundary conditions (called liner) has also important industrial applications, e.g., lined nacelles of an aircraft engine, ventilating systems, underwater acoustics *etc*.

There are an infinite number of modes in a WIBC. They can be classified in two categories[3, 4, 5] : guided modes resulting from the finiteness of the waveguide geometry, and surface modes that exist only near the waveguide wall and decay exponentially away from the wall when impedance is spring like. A typical eigenvalue distribution for a cylindrical WIBC is shown in Fig. 1[6] when $K = 30, \beta_0 = 0.4 + 0.2j$ which are typically industrial values in the lined intakes of an aeroengine. There is only one surface mode when $m = 0$ as shown in Fig. 1 (upper panel). There are an infinite number of discrete surface modes in a cylindrical WIBC corresponding to $m = 0 - \infty$, as shown in Fig. 1(a) by "$\oplus$". For each azimuthal order $|m|$ (except $m = 0$), there are only two ($+|m|$ and $-|m|$) surface modes which are in degeneracy. It is noted that this degeneracy is totally different from the branch points and exceptional points in the following sections. In the lower panel (a) of Fig. 1, each $\oplus$ corresponds to one $|m|$. They are arranged as $m = 0, \quad \pm 1, \quad \pm 2, \quad \cdots$, from left to right. The decaying rates of the surface mode amplitudes away from the wall are decided by the imaginary parts of the surface mode eigenvalues γ_m. A typical surface mode profile corresponding to $m = 2$ is shown in the lower panel (c) and (d) of Fig. 1. It needs to stress that the surface modes in a WIBC are asymptotic solutions in high frequency ω. The eigenfunctions become exponentially decaying along r like $e^{\omega|\Im m(\gamma)|(1-r)}/\sqrt{r}$,[5] where $\Im$m refers to the imaginary part. Strictly speaking, they should be called "quasi-surface modes". The eigenvalues of guided modes are marked by "o" in the lower panel (a) of Fig. 1. The eigenfunction of guided mode $(2, 1)$, as an example, is plotted in the lower panel (b) of Fig. 1.

There exist double eigenvalues in WIBC, which was first inquired by Morse[7], and then studied in detail by Tester[8, 9], Zorumski *et al*[10], Shendrov[11]. It has been shown that the corresponding impedances are square root branch point in the complex admittance plane. Recently, the corresponding coalescences of eigenfunctions at the branch points and their important effects on the sound propagation has been shown by Bi and Pagneux[6]. It needs to point out that Tester[8, 9] and Mechel [3, 4] linked the branch points with the Creamer's optimum impedance. Creamer's optimum impedance in an infinite WIBC, proposed firstly

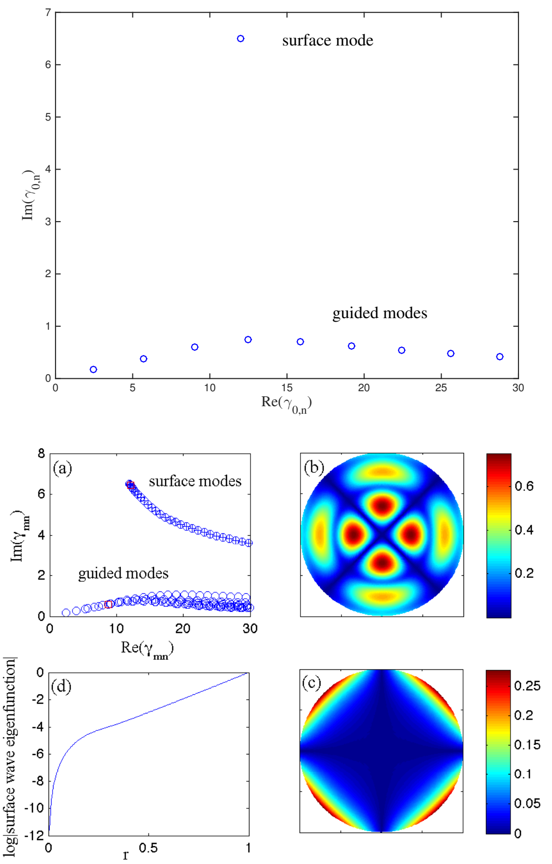

FIGURE 1 – Typical eigenvalues and eigenfunctions in a WIBC.
Upper panel : eigenvalues when $m = 0$.
Lower panel : (a) eigenvalues when $|m| = 0 - 30$. $\oplus$ refers to surface modes (eigenvalues corresponding to $\text{Im}(\gamma_{mn}) > 3$ in this figure), (b) eigenfunction (not normalized) of guided mode (2,1), whose eigenvalue is shown as $\square$ in the branch of guided modes in (a), (c) eigenfunction (not normalized) of surface mode $m = 2$, whose eigenvalue is shown as $\square$ in the branch of surface modes in (a), (d) the eigenfunction profile along r of surface mode $m = 2$. $K = 30, \beta_0 = 0.4 + 0.2j$.

by Cremer[12] is an impedance at which the maximum attenuation of the least attenuation mode achieves. It has been one of the most important liner design method, e.g., [13, 14, 15, 16, 17, 18, 19, 20]. Tester[8, 9] argued that not only the Cremer's optimum impedance might be corresponding to a branch point, but also for any pair of neighbour modes, the corresponding branch points might be the optimum impedance at which one of the mode achieve maximum attenuation. We have carried out numerous calculations and observed similar conclusions[6].

The mechanism of the possibly maximum attenuations at branch points is not explained to date. As was pointed by Tester[8] in 1973 that "A most intriguing property of theoretical and experimental decay rates of modes in lined ducts, for which there is no obvious explanation, is the existence of maximum decay rates for values of the liner impedance which, at first sight, are arbitrary and totally unconnected with any simple results associated with absorption by reflecting boundaries.". This work is motivated by Tester's curious question.

Resonance trapping phenominon in open quantum system has been recently observed in an open microwave cavity[21]. With increasing coupling strength to the continuum of decay channels, only a few resonances (modes) align with the open channels and become short lived (the widths of resonances, or the imaginary parts of the eigenvalues of mode, increase), while the widths of the remaining resonances (modes) first increase but finally decrease again and become long lived (trapped modes). The decoupling of some resonances (modes) from the open channels takes place, although the coupling strengths increase, and different times scales appear.

2 Model

We consider an infinite long cylindrical waveguide, of uniform and circular cross section, having locally reactive impedance wall boundary conditions. The impedance is assumed uniform along axial and circumferential directions, respectively. Linear and lossless sound propagation in air is assumed. With time dependence $\exp(j\omega t)$ omitted, the eigenvalues γ and eigenfunctions ϕ of modes satisfies the Laplacian eigenvalue problem

$$\nabla_{\perp}^2 \phi_{mn} = -\gamma_{mn}^2 \phi_{mn}, \tag{1}$$

where

$$\nabla_{\perp}^2 = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}, \tag{2}$$

with the boundary condition

$$\frac{\partial \phi_{mn}}{\partial r} = Y\phi_{mn}, \quad \text{at} \ \ r = 1, \tag{3}$$

where m and n refer to, respectively, the circumferential and radial mode indices. $Y = -jK\beta_0$. $\beta_0 = 1/Z_0$, where Z_0 and β_0 are wall boundary impedance and admittance, respectively. They are complex number. $K = \omega R/c_0$ refers to the dimensionless frequency, R is the radius of the waveguide. By assuming the solution

$$\phi_{mn}(r,\theta) = \frac{J_m(\gamma_{mn} r)}{J_m(\gamma_{mn})}\left\{ \begin{array}{l} \cos(m\theta) \\ \sin(m\theta), \end{array}\right. \tag{4}$$

we obtain the dispersion equation for the eigenvalues

$$\gamma_{mn}\frac{J_m'(\gamma_{mn})}{J_m(\gamma_{mn})} = Y. \tag{5}$$

Equation (5) has infinitely complex solutions γ. It is difficult to solve it without missing solutions. We use the method developed in Ref. [22] which can export all ($M \times I$, where M and I refer to the truncation numbers for indice m and i, respectively, see below) eigenvalues and eigenfunctions at a time without iteration and missing solutions. We expand the eigenfunctions $\phi_{m,n}$ in WIBC in terms of the eigenfunctions $\psi_{m,i}$ of an infinite waveguides with rigid boundary conditions $\partial\psi_{mi}/\partial r = 0$,

$$\phi_n = \sum_{i=0}^{I} c_{n,i}\psi_i(r,\theta) = \psi^T \mathbf{c}, \tag{6}$$

where I is the truncation of the expansion in radial direction. For simplicity, we have dropped in Eq. (6) the circumferential index m because of non-coupling in this direction in this section, i.e., we consider only the problem in radial direction. By projecting the Eq. (1) over the base ψ_i, using the boundary condition (3), we obtain a matrix eigenvalue problem[22]

$$\mathsf{H}\mathbf{c}_n = \gamma_n^2 \mathbf{c}_n, \quad \mathit{where}, \mathsf{H} = \mathsf{H}_0 + jK\beta_0 \mathsf{H}_1, \tag{7}$$

where H_0 and H_1 are real and symmetric matrices (Hermitian). H_0 is a diagonal matrix, its elements in the main diagonal are the eigenvalues of rigid modes α_n^2. $\mathsf{H}_1 = \mathbf{c_s}\mathbf{c_s}^T$, where "^{T}" refers to transpose, $\mathbf{c_s}$ is a column vector, its elements are $\psi_i(r = 1)$. Using Eq. (7), the eigenvalue problem in a waveguide with impedance boundary conditions may be interpreted, by analogy to the resonance in an open quantum system, as a close system, i.e., the rigid waveguide represented by matrix H_0 opened to or interacted with an environment, i.e., here represented by matrix H_1. The $K\beta_0$ may be interpreted as coupling strength between the close system and the environment. For each m, H_1 is a rank 1 matrix, its eigenvalue and eigenvector are

$$\mathsf{H}_1\mathbf{c_s} = \mathbf{c_s}^T\mathbf{c_s}\mathbf{c_s}, \tag{8}$$

where $\mathbf{c_s}^T\mathbf{c_s}$ is the eigenvalue. It needs to stress that although H_1 analogy to the environment, i.e., one open channel in open quantum system, however, its elements are the rigid eigenfunctions at the wall, therefore it represent a close environment, the rigid boundary wall of the waveguide.

3 Branch points and Exceptional points

The dispersion Eq.(5) exist infinite double roots corresponding to

$$\gamma_{mn}\frac{J_m'(\gamma_{mn})}{J_m(\gamma_{mn})}\Big|_{\gamma_{mn}=\gamma_{BP}} = -jK\beta_{BP}, \tag{9}$$

$$\frac{\partial}{\partial\gamma_{mn}}\left(\gamma_{mn}\frac{J_m'(\gamma_{mn})}{J_m(\gamma_{mn})}\right)\Big|_{\gamma_{mn}=\gamma_{BP}} = 0, \tag{10}$$

In the vicinity of the double eigenvalues, the eigenvalues, which have no power series expansion, are expressed approximately to the lowest order as

$$\gamma_n - \gamma_{BP} \approx -\sqrt{\frac{2\partial f/\partial\beta_0}{\partial^2 f/\partial\gamma_n^2}}\sqrt{\beta_0 - \beta_{BP}}, \tag{11}$$

where we have assumed that the dispersion equation (5) has no triple or higher order eigenvalues, β_{BP} refers to the admittance at which the double eigenvalues occur and the function f is

$$f(\gamma_{mn}, \beta_0) = \gamma_{mn}\frac{J_m'(\gamma_{mn})}{J_m(\gamma_{mn})} + jK\beta_0. \tag{12}$$

Equation (11) shows clearly that β_{BP} are square root branch points in the complex value admittance plane. Only when the admittance is spring-like, i.e., $\Im m(\beta_0) > 0$ (convention $e^{j\omega t}$ is used), there exist branch pints.

The physical reality of square root branch point behaviour has been experimentally observed by Dembowski *et al*[23] in a microwave cavity with dissipation, recently. This physical reality can also be proved in waveguides with impedance boundary conditions as proposed in Ref. [6]. At the branch points, not only the eigenvalues of a pair of neighbour modes, but also the corresponding eigenfunctions coalesce, the left and right eigenfunctions of the coalescent modes are orthogonal (self-orthogonality)[6].

The points in a complex plane at which both eigenvalues and the corresponding eigenfunctions coalesce is called exceptional points (EPs). EPs should not be confused with a degeneracies, as mentioned above for the surface modes of $+|m|$ and $-|m|$, at which the corresponding eigenfunctions are still orthogonal. Recently, EPs have attracted much attention. The important properties of EPs have been uncovered by Heiss[24, 25, 27, 26], Rotter[28], and Berry[29] for physical systems with dissipation or non-Hermitian system. EPs have been found in different systems, such as, laser-induced ionization states of atoms [30], electronic circuits [31], atoms in cross magnetic and electric fields [32], a chaotic optical microcavity[33], and PT-symmetric waveguides[34]. The effects of EPs in acoustics have been developed recently by Bi and Pangneux[6] and Xiong *et al*[35].

There are an infinite number of EPs in the complex admittance plane for each circumferenial index m[6]. The first 10 EPs when $m = 0$ are illustrated in the upper panel of Fig. 2. The EPs separate the complex admittance plane into two regions : in the lower region, there exist only guided modes, whereas in the upper region, there exist guided modes and one surface mode (for each m). The surface modes exist only when the admittance is spring-like, i.e., $\Im m(\beta_0) > 0$ (convention $e^{j\omega t}$ is used).

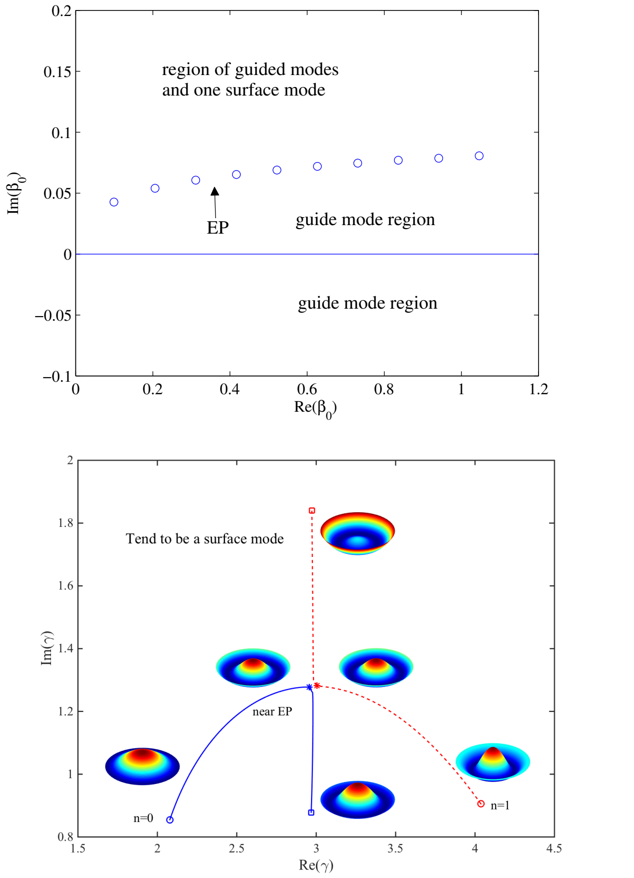

FIGURE 2 – Upper panel panel : Distribution of the first 10 EPs in the complex admittance plane, when $m = 0$. Lower panel : Eigenvalue trajectories passing near the first EP as a function of $\Im m(\beta_0)$. $\Re e(\beta_0) = 0.09935 > \Re e(\beta_{EP})$. $\Im m(\beta_0) = 0 - 0.05$. '$\circ$', $\Im m(\beta_0) = 0$; '$\Box$', $\Im m(\beta_0) = 0.05$; and '$*$' refers to near $\Im m(\beta_{EP})$. $m = 0$.

4 Resonance trapping near an EP

We first consider the resonance trapping near one EP, e.g., the first EP ($\beta_{EP} = 0.099346 + 0.042653j$) in the upper panel of Fig. (2). The eigenvalue trajectories in the vicinity of the first EP is shown in the lower panel of Fig. 2 as a function of $\Im m(\beta_0)$, when $\Re e(\beta_0) = 0.09935$ is fixed. The eigenfunctions at some selected β_0 are also plotted. As $\Im m(\beta_0)$ increase, the imaginary parts of the eigenvalues of mode $n = 0$ and those of mode $n = 1$ increase until β_0 approaches the EP, where the eigenvalues form an avoided crossing and the eigenfunctions mix strongly. With a further increase of $\Im m(\beta_0)$, the imaginary part of mode $n = 1$ continue to increase to turn to be a localised mode which is localise near the waveguide wall, while the imaginary part of mode $n = 0$ decreases and turn to a mode which resembles mode $n = 1$ with a small imaginary part. This process is very similar to the resonance trapping in open quantum systems. However, because the environment is close, the mode with larger imaginary part which aligns with the environment is localised near the wall to form a quasi-surface mode.

Here we stress the necessary of EPs for the resonance trapping, which has been overlooked in the observations of resonance trapping in the literature[21]. This is more clear if we consider the impedance boundary is mass-like, or the coupling strength is in form $K\beta_0 = K(\Re e(\beta_0) - j|\Im m(\beta_0)|)$, in which EPs do not exist, resonance trapping and localised modes do not exist either.

5 Resonance trapping as a global behaviour

When the $K|\beta_0|$ is small, $|\beta_0| \ll |\beta_{EP}|$, the presence of impedance at the wall is only a perturbation of the rigid waveguide. As $|\beta_0|$ increase from 0, the imaginary parts of the eigenvalues increase from the eigenvalues of rigid modes (modes in a waveguide with boundary condition $\beta_0 = 0$) and the real parts of the eigenvalues shift to these of soft modes (modes in a waveguide with impedance $\mathbb{Z}_0 = 0$) as shown in Fig. 3.

On the other hand, as $K\Im m(\beta_0) \gg K\Re e(\beta_0) > \Im m(\beta_{EP})$, $\mathsf{H} = \mathsf{H}_0 + jK\beta_0\mathsf{H}_1 \approx jK\beta_0\mathsf{H}_1 = jK\beta_0\mathbf{c_s}\mathbf{c_s}^T$. Therefore, Eq. (7) is rewritten as

$$\mathsf{H}\mathbf{c_s} \approx jK\beta_0\mathsf{H}_1\mathbf{c_s} = jK\beta_0\mathbf{c_s}^T\mathbf{c_s}\mathbf{c_s} = \gamma_s^2\mathbf{c_s}. \qquad (13)$$

It means that the only eigenvalue is

$$\gamma_s = \sqrt{jK[\Re e(\beta_0) + j\Im m(\beta_0)]\mathbf{c_s}^T\mathbf{c_s}} \qquad (14)$$

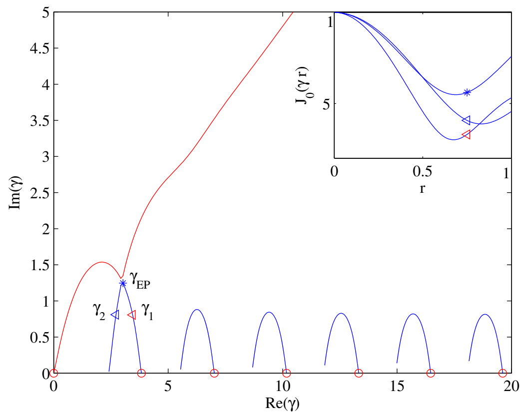

FIGURE 3 – Eigenvalue trajectories passing near the first EP as a function of $|\beta_0|$. $arg(\beta_0) = atan(0.042653/0.099347)$. $|\beta_0| = 0 - 100/30$. $\gamma_1 = 3.4488 + 0.8076j$, $\gamma_2 = 2.7219 + 0.80957j$, $\gamma_{EP} = 3.0285 + 1.2467j$, $m = 0$. Inset, the eigenfunctions along r corresponding to γ_1, γ_2 and γ_{EP}, respectively.

$$= j\sqrt{K\Im m(\beta_0)\mathbf{c_s}^T\mathbf{c_s}}\sqrt{1 - j\frac{\Re e(\beta_0)}{\Im m(\beta_0)}}$$

$$\approx \frac{\Re e(\beta_0)}{2}\sqrt{\frac{K\mathbf{c_s}^T\mathbf{c_s}}{\Im m(\beta_0)}} + j\sqrt{K\mathbf{c_s}^T\mathbf{c_s}\Im m(\beta_0)}.$$

The eigenfunction is

$$\phi_s \propto |\frac{J_m(\gamma_s r)}{J_m(\gamma_s)}| \propto \frac{e^{\sqrt{K\mathbf{c_s}^T\mathbf{c_s}\Im m(\beta_0)}(1-r)}}{\sqrt{r}}. \tag{15}$$

It represents a quasi-surface mode near the wall, decay exponentially away from the wall, produced by the presence of impedance wall, when $\Im m(\beta_0) > 0$, i.e. spring-like impedance. Its eigenvalue has very large imaginary part $\sqrt{K\mathbf{c_s}^T\mathbf{c_s}\Im m(\beta_0)}$. It is evident from Eq. (14) that when $\Im m(\beta_0) < 0$, modes with very large imaginary part will not be possible when $|\Im m(\beta_0)|$ continually increase. Except of the surface mode ϕ_s (eigenvector $\mathbf{c_s}$), matrix H has another $I-1$ eigenvalues γ_n and eigenvectors $\mathbf{c}_n$, $n \neq s$, where I refers to the truncation of the expansion (6). The I eigenvectors are bi-orthogonal, e.g. $\mathbf{c_s}^T\mathbf{c}_n = 0$, $n \neq s$. Therefore, when $n \neq s$ and using Eq (8)

$$\mathsf{H}\mathbf{c}_n = \mathbf{H}_0\mathbf{c}_n + jK\beta_0\mathsf{H}_1\mathbf{c}_n \approx \mathbf{H}_0\mathbf{c}_n = \alpha_n^2\mathbf{c}_n, \tag{16}$$

where we have used the orthogoal relation $\mathbf{c_s}^T\mathbf{c}_n = 0$ ($n \neq s$). Eq (16) means that when we increase $K\Im m(\beta_0)$, i.e., $K\Im m(\beta_0) \gg K\Re e(\beta_0) > \Im m(\beta_{EP})$, except for the surface mode, with very large imaginary part of eigenvalue, all the other modes return to be the rigid modes of the waveguides. This surprising behavior, called 'resonance trapping', has also been found in an open microwave cavity[21]. In order to distinguish from the resonance trapping near an EP, we call it global resonance trapping.

The above analytical analysis can be shown, by an numerical example in Fig. 3 in which we plot the eigenvalue trajectories as a function of $|\beta_0|$ when the phase $arg(\beta_0) = atan(0.042653/0.099347)$. This phase corresponds to the phase of the first EP β_{EP}. In the inset, we plot the eigenfunctions along r for different γ ($|\beta_0|$). It is clearly to show that with increasing the $|\beta_0|$, the imaginary part of the eigenvalue increases from zero to that of γ_1, then reach the maximum γ_{EP} when $\beta_0 = \beta_{EP}$. The eigenfunction at wall increases and also reach the maximum, it means that sound field is pushed towards to the wall which is easier to be absorbed. With a further increase of $|\beta_0|$, the imaginary part of the eigenvalue *decreases*, the eigenfunction at wall *decease*, which means that sound field is pushed away from the wall which is more difficult to be absorbed.

It needs to stress that the global resonance trapping occur only when the admittance is spring-like, i.e., $\Im m(\beta_0) > 0$ (convention $e^{j\omega t}$ is used). The necessary conditions : the admittance is spring-like, i.e., $\Im m(\beta_0) > 0$ for the exist of EPs, localisation modes, and the global resonance trapping suggest the essential roles of EPs for the presence of resonance trappings.

6 Conclusion

We have studied an analogy of resonance trapping in open quantum to modes in an acoustic close system - a Waveguide with Impedance (admittance) Boundary Conditions (WIBC). By projecting the eigenvalue problem of the WIBC onto the corresponding rigid mode basis, we obtain an eigenvalue problem of matrix $\mathsf{H} = \mathsf{H}_0 + jK\beta_0\mathsf{H}_1$. The mode behaviour of the WIBC may be interpreted as interacting between the modes of corresponding rigid waveguide with an environment described by H_1, and the admittance plays the roles of coupling strength. With increasing the imaginary part of $K\beta_0$ (coupling strength), one mode align with the wall of the waveguide (the environment represented by H_1) and become localization, while the imaginary parts of eigenvalues of the remaining modes first increase but finally decrease again and return nearly to be rigid modes. We show that the exists of (Exceptional Points) EPs are the essential ingredients for the occurs of resonance trapping. Our analogy provides a new insight into resonance trapping and provide a new point of view for understanding the mode behaviours in waveguides.

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